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DEVELOPMENT OF LIQUID FLOW IN A PLANE CHANNEL WITH MOVING PERMEABLE WALLS

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We determined the laws that govern the change in hydraulic resistance in viscous incompressible liquid flow between parallel permeable and moving walls. The laws are of great practical interest.

We consider a viscous incompressible liquid flow between parallel permeable walls moving in their planes with prescribed constant velocities. The study and determination of the regularities of such motions are of both theoretical and great practical interest because of various technical applications: control of a boundary layer, variation of the flow velocity in tubes by injection or suction of liquid, study of the specific features of rivulets flowing in soil, etc.

1. We assume the liquid motion to be laminar, stationary, plane-parallel, and isothermal, with no body forces. The permeable walls of the channel move in their planes in the direction of the $0x$ axis with constant velocities U_1 and U_2. There is also a forced flow in the channel that has a plane uniform velocity profile at the inlet (Fig. 1). The initial equations of liquid motion are considered to be the following equations obtained from the Navier–Stokes system [1]:

$$U\frac{\partial v_x}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 v_x}{\partial y^2}, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{1.1}$$

Assuming the coordinate origin to be located on the middle plane of the channel, the boundary conditions of the problem can be written in the form:

$$\begin{aligned} &\text{if } x = 0, \ |y| \le h, \text{ then } v_x = U, \ p = p_{\text{in}}; \\ &\text{if } x > 0, \ y = h, \text{ then } v_x = U_1, \ v_y = k(p - p_{\text{ex}}); \\ &\text{if } x > 0, \ y = -h, \text{ then } v_x = U_2, \ v_y = -k(P - p_{\text{ex}}). \end{aligned} \tag{1.2}$$

When $p > p_{\text{ex}}$, there is liquid suction; when $p < p_{\text{ex}}$, there is injection of liquid. With allowance for the new variables

$$z = \frac{x}{h}, \quad \eta = \frac{y}{h}, \quad u = \frac{v_x - U}{U}, \quad v = \frac{v_y}{U}, \quad p = \frac{p - p_{\text{in}}}{\rho U^2} \tag{1.3}$$

system of Eqs. (1.1) and boundary conditions (1.2) can be rewritten in the form

$$\frac{\partial u}{\partial z} = -\frac{\partial P}{\partial z} + \frac{1}{\text{Re}}\frac{\partial^2 u}{\partial \eta^2}, \quad \frac{\partial P}{\partial \eta} = 0, \quad \frac{\partial u}{\partial z} + \frac{\partial v}{\partial \eta} = 0, \tag{1.4}$$

where $\text{Re} = Uh/\nu$ is the Reynolds number;

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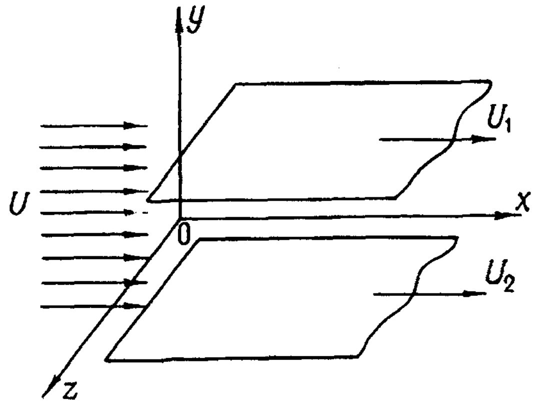


Fig. 1. Scheme of problem statement.

$$\text{if } z=0, \text{ then } u=0, \ P=0;$$

$$\text{if } \eta=1, \ z>0, \text{ then } u=\frac{U_1-U}{U}, \ v=\alpha(P+b); \qquad (1.5)$$

$$\text{if } \eta=-1, \ z>0, \text{ then } u=\frac{U_2-U}{U}, \ v=-\alpha(p+b),$$

where $\alpha = k\rho u$; $b=(p_{\text{in}}-p_{\text{ex}})/\rho U^2$.

2. Applying the Laplace integral transform to Eqs. (1.4) and boundary conditions (1.5), we obtain

$$\frac{1}{\text{Re}}\frac{d^2\bar{u}}{d\eta^2}-\lambda\bar{u}=\lambda\bar{P}, \quad \frac{\partial\bar{P}}{\partial\eta}=0, \quad \frac{d\bar{v}}{d\eta}+\lambda\bar{u}=0, \qquad (2.1)$$

where

$$\bar{u}=\int_0^\infty u\exp(-\lambda z)\,dz, \quad \bar{P}=\int_0^\infty P\exp(-\lambda z)\,dz,$$

$$\bar{v}=\int_0^\infty v\exp(-\lambda z)\,dz,$$

and λ is the transformation parameter;

$$\text{if } \eta=1, \ z>0, \text{ then } \bar{u}=\frac{U_1-U}{U}\frac{1}{\lambda}, \ \bar{v}=\alpha\left(\bar{P}+\frac{b}{\lambda}\right);$$

$$\text{if } \eta=-1, \ z>0, \text{ then } \bar{u}=\frac{U_2-U}{U}\frac{1}{\lambda}, \ \bar{v}=-\alpha\left(\bar{P}+\frac{b}{\lambda}\right). \qquad (2.2)$$

The solutions of system of equations (2.1) subject to boundary conditions (2.2) are

$$\bar{u}=\frac{(U_1-U_2)\,\text{sh}\,\beta\eta}{2\lambda U\,\text{sh}\,\beta}+\frac{\text{ch}\,\beta\eta\,[B(\beta_1-\beta^2)-2b\beta_1]}{2\lambda\,\text{ch}\,\beta\,(\beta\,\text{th}\,\beta-\beta^2+\beta_1)}-\bar{P}, \qquad (2.3)$$

$$\bar{P}=-\frac{B\beta\,\text{th}\,\beta+2b\beta_1}{2\lambda(\beta\,\text{th}\,\beta-\beta^2+\beta_1)}, \qquad (2.4)$$

$$\bar{v} = \frac{(U_1 - U_2)(\operatorname{ch}\beta - \operatorname{ch}\beta\eta)}{2\beta U \operatorname{sh}\beta} - \frac{\operatorname{sh}\beta\eta\,[B(\beta_1 - \beta^2) - 2b\beta_1]}{2\beta \operatorname{ch}\beta(\beta \operatorname{th}\beta - \beta^2 + \beta_1)} + \lambda\eta\bar{P}, \tag{2.5}$$

where

$$B = \frac{U_1 + U_2}{U} - 2;\ \beta_1 = \alpha\,\mathrm{Re};\ \beta^2 = \lambda\,\mathrm{Re}.$$

Making the inverse Laplace transformation and passing to the former variables by formulas (1.3), for the unknown functions v_x, v_y, and p we finally obtain

$$v_x = \frac{U_1 - U_2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \sin \pi n y/h}{n}\exp\left(-\frac{n^2\pi^2 x}{\mathrm{Re}\,h}\right) +$$

$$+ B_1 U\left(1 - \frac{\operatorname{ch}\sqrt{\lambda_1 \mathrm{Re}}\,y/h}{\operatorname{ch}\sqrt{\lambda_1\mathrm{Re}}}\right)\exp(\lambda_1 x/h) +$$

$$+ U\sum_{m=1}^{\infty} A_m\left(1 - \frac{\cos\gamma_m y/h}{\cos\gamma_m}\right)\exp\left(-\frac{\gamma_m^2 x}{\mathrm{Re}\,h}\right) + \frac{U_1}{2}\left(1 + \frac{y}{h}\right) + \frac{U_2}{2}\left(1 - \frac{y}{h}\right), \tag{2.6}$$

$$v_y = \frac{U_2 - U_1}{\mathrm{Re}}\sum_{n=1}^{\infty}\left[(-1)^n\cos\frac{\pi n y}{h} - 1\right]\exp\left(-\frac{\pi^2 n^2 x}{\mathrm{Re}\,h}\right) -$$

$$- B_1 U\lambda_1\left(\frac{y}{h} - \frac{\operatorname{sh}\sqrt{\lambda_1\mathrm{Re}}\,y/h}{\operatorname{sh}\sqrt{\lambda_1\mathrm{Re}}}\right)\exp\left(\frac{\lambda_1 x}{h}\right) + \frac{U}{\mathrm{Re}}\sum_{m=1}^{\infty}A_m\gamma_m^2\left(\frac{y}{h} - \frac{\sin\gamma_m y/h}{\sin\gamma_m}\right)\exp\left(-\frac{\gamma_m^2 x}{\mathrm{Re}\,h}\right), \tag{2.7}$$

$$p = p_{\mathrm{ex}} - B_1\rho U^2\exp(\lambda_1 x/h) - \rho U^2\sum_{m=1}^{\infty}A_m\exp\left(-\frac{\gamma_m^2 x}{\mathrm{Re}\,h}\right). \tag{2.8}$$

Here

$$B_1 = \frac{\beta_1(B - 2b) - \lambda B\,\mathrm{Re}}{(\lambda_1\mathrm{Re})^2 - 2\beta_1\lambda_1\mathrm{Re} + \beta_1^2 + \beta_1};\quad A_m = \frac{B\gamma_m^2 + \beta_1(B - 2b)}{\gamma_m^4 + 2\beta_1\gamma_m^2 + \beta_1^2 + \beta_1};$$

λ_1 is the value corresponding to the two real roots of the equation $\operatorname{th}\sqrt{\lambda\mathrm{Re}} = \sqrt{\lambda\mathrm{Re}} - \alpha\mathrm{Re}/\sqrt{\lambda\mathrm{Re}}$, and γ_m are the real roots of the equation $\tan\gamma = \gamma + \alpha\mathrm{Re}/\gamma$.

Substituting the first two terms of the expansion $\tan\beta = \beta - 1/3\beta^3$ into Eqs. (2.3)-(2.5) and performing all the mathematical computations, we obtain the values of the inverse transforms at a sufficient distance from the entrance:

$$v_{x\infty} = \frac{3U}{4}\left(1 - \frac{y^2}{h^2}\right)\left[\frac{k\rho U(B - 2b)}{\lambda_2}\operatorname{sh}\lambda_2 x/h - B\operatorname{ch}\lambda_2 x/h\right] +$$

$$+ \frac{U_1 - U_2}{2h}y + \frac{U_1 + U_2}{2}, \tag{2.9}$$

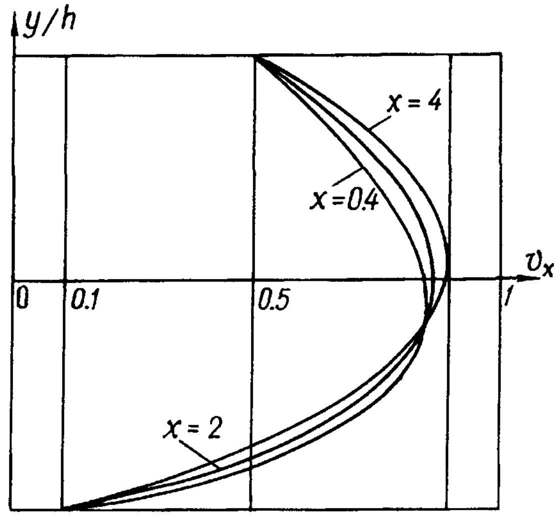


Fig. 2. Change in the longitudinal velocity component along the length and over the height in the initial section of the channel. v_x, m/sec.

$$v_{y\infty} = \frac{3U}{4}\left(\frac{y^3}{3h^3} - \frac{y}{h}\right)[k\rho U(B-2b)\,\mathrm{ch}\,\lambda_2 x/h - B\lambda_2\,\mathrm{sh}\,\lambda_2 x/h], \tag{2.10}$$

$$p_\infty = p_{ex} + \frac{3B\mu U}{2\lambda_2 h}\,\mathrm{sh}\,\lambda_2 x/h - \frac{\rho U^2(B-2b)}{2}\,\mathrm{ch}\,\lambda_2 x/h, \tag{2.11}$$

where $\lambda_2^2 = 3\mu k/h = 3\alpha/\mathrm{Re} = \lambda_1^2$.

We will calculate the length of the percolation region (l), i.e., the region of the channel over which the internal pressure of the liquid (p) is larger than the external one (p_{ex}). Assuming p_∞ to be equal to p_{ex} in Eq. (2.10), we obtain for length l the following approximate value:

$$l = \frac{h}{\lambda_2}\,\mathrm{arcth}\,\frac{U(B-2b)\lambda_2 h}{3B\nu}. \tag{2.12}$$

For a viscous liquid flow in tubes of great practical value is the coefficient of hydraulic resistance, which in the case of immovable impenetrable walls depends only on the dissipation of the mechanical energy of the liquid due to the work done by internal friction forces. Usually this dependence is determined by the formula

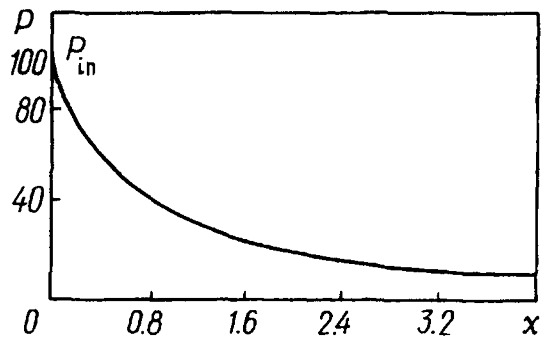
$$\frac{\partial p}{\partial x} = -\xi\frac{\rho U_{\mathrm{mean}}}{2}\frac{1}{2h}. \tag{2.13}$$

In the case of moving and permeable walls, the hydraulic resistance coefficient is connected to the external (with respect to the liquid) friction forces; therefore, it can take any values (zero and even negative ones). In the case of moving walls, the value of ξ was obtained by us in [2].

To determine the coefficient ξ for a developed flow of a viscous liquid in a channel with moving and permeable walls, we determine $\partial p/\partial x$ from Eq. (2.8):

$$\frac{\partial p}{\partial x} = -\frac{B_1\lambda_1}{h}\rho U^2\exp\left(\frac{\lambda_1 x}{h}\right) + \frac{\rho U^2}{\mathrm{Re}\,h}\sum_{m=1}^{\infty} A_m\gamma_m^2\exp\left(-\gamma_m^2\frac{x}{\mathrm{Re}\,h}\right). \tag{2.14}$$

Equations (2.13) and (2.14) yield



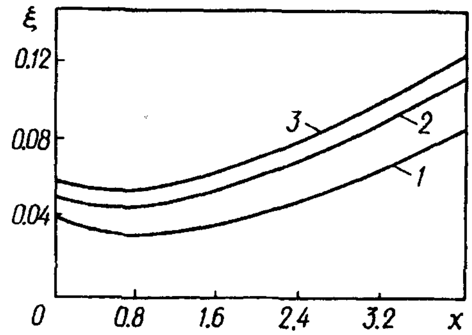


Fig. 3. Change of pressure along the channel. p, kg/m^2.

Fig. 4. Change in the hydraulic resistance coefficient: 1) moving walls; 2) one wall is fixed; 3) both walls are fixed.

$$\xi = 4B_1\lambda_1 \exp(\lambda_1 x/h) - \frac{4}{\mathrm{Re}} \sum_{m=1}^{\infty} A_m \gamma_m^2 \exp\left(-\frac{\gamma_m^2 x}{\mathrm{Re}\, h}\right). \tag{2.15}$$

Equations (2.6)-(2.8) and (2.15) describe the laws of the change in velocity, pressure, and resistance coefficient, respectively, in a developed viscous incompressible liquid flow in a plane channel with moving and permeable walls.

Assigning different values to the characteristic parameters U, U_1, U_2, and k, we obtain all possible specific cases in investigation of the development of viscous liquid motion in the above statement of the problem. Thus, e.g., assuming $k = 0$ in Eqs. (2.6)-(2.14), we find the unknown quantities [3]. In the case of $U_1 = U_2 = 0$ the results coincide with their values obtained in [4].

We will consider a numerical example with the following data: $U = 1$ m/sec, $U_1 = 0.5$ m/sec, $U_2 = 0.1$ m/sec, $h = 0.05$ m, $\nu = 10^{-4}$ m^2/sec, $P_{in} = 100$ kg/m^2, $\rho = 102$ kg·sec^2/m^4, $\alpha = 10^{-5}$.

The changes in the axial velocity v_x along the length and over the height of the channel are shown in Fig. 2. From the figure it is seen that in the upper portion of the channel from the middle line ($y = 0$) v_x increases along the line, and in the lower portion it decreases; moreover, in different sections the maximum value lies at different heights.

The change in pressure along the channel is shown in Fig. 3. Figure 4 presents the law of the change in the friction resistance coefficient ξ. It is seen that in the case of fixed permeable walls the value of ξ is greater than for moving permeable walls (Fig. 4, curve 3). If both permeable walls move, then ξ is smaller than in the case of one moving wall (Fig. 4, curve 2). One observes an interesting trend: the permeability decreases friction in the case of moving walls and, conversely, increases it in the case of fixed walls. One other feature is worthy of note. In the case of impermeable walls (moving, fixed) the friction resistance coefficient decreases along the channel [3], whereas with permeable (moving, fixed) walls it increases.

For the data selected, the length of the percolation region calculated by formula (2.15) is approximately equal to 20 m.

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NOTATION

U, cross-sectional mean velocity of forced flow at the beginning of the channel; U_1, U_2, velocities of the walls; v_x, v_y, components of flow velocity; p, pressure; ρ, density; ν, kinematic viscosity of liquid; p_{in}, pressure in the initial section of the channel; p_{ex}, external pressure; $2h$, distance between the walls; k, coefficient of the permeability of the walls; λ, Laplace transformation parameter; Re, Reynolds number; ξ, hydraulic resistance coefficient; m, n, positive integers; l, length of the percolation region.

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